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Essential Mathematics for Undegraduate Students in Engineering

By

Given a function (x)

where p is the static fluid pressure, is fluid density, V is velocity, g is acceleration due to gravity and h is vertical height. In the first form, all terms have dimensions of pressure (force per unit area), while in the second form, all terms have dimensions of length.

C. Differentials

Often you will come across expressions involving differential quantity such as:

$$dy = \frac{dy}{dx}dx \qquad (3)$$

in which it appears at first glance that the dx's have cancelled each other out. A more useful way to think about the meaning of (3) is:

change in y (or dy) = (rate of change of y with respect to x) times (change in x) (4)

One can extend this notion and write

$$y(x + dx) = y(x) + \frac{dy}{dx}dx \qquad (5)$$

which states that the value of y at x+dx is given by its value at x (the first term on the right hand side of (5)) plus (the rate of change of y with respect to changes in x) times (change in x, namely dx). Note that dimensions of all terms in (5) also work out correctly. Can you figure out $y(x + dx/2)$ and $y(x - dx/2)$? Simply replace "+dx" in (5) by "+dx/2" or "-dx/2" to get the result. Application of this concept is usually tied to a neatly labeled sketch, in which the axis (x-axis in this case) is clearly indicated, with the arrowhead denoting the direction of increasing x.

The second derivative of y with respect to x is denoted as $\frac{d^2y}{dx^2}$ or $y''(x)$. The prime notation is useful for first, second, and perhaps third derivative, but is often not used beyond that. If it is, for example, the n-th derivative of y, it is denoted by $y^{(n)}(x)$.

What are the dimensions of $\frac{d^2y}{dx^2}$? Answer: $\frac{d^2y}{dx^2} \sim \frac{y}{x^2}$. If y has units of meters, and x is in seconds, the quantity $\frac{d^2y}{dx^2}$ denotes acceleration with units of m/s^2.

D. Chain Rule

If $y = g(f(x))$, then the derivative

$$\frac{dy}{dx} = \frac{dg}{dx} = \frac{dg}{df}\frac{df}{dx} \qquad (6)$$

Note that the quantity $\frac{dg}{df}$ denotes the rate of change of g

Note that $\frac{dy}{dx}$ denotes the rate at which y changes with respect to changes in x. A quick check of dimensions in (7) reveals that:

$$\frac{y}{x} \quad \frac{y}{h}\frac{h}{f}\frac{f}{x} \qquad (8)$$

Note that the dimensions of h and dimensions of f cancel out on the right hand side of (8) as they appear in both the numerator and denominator, leaving the dimensions of y in the numerator and that of x in the denominator. Master the concept of chain rule and it will serve you well in engineering/physics courses.

<u>E. Polynomials</u>

The simplest polynomial is a constant, also considered as a polynomial of degree 0. Thus,

$$P_0(x) \quad c \qquad (9)$$

The next, in terms of simplicity, is a linear function of x. It can be written as

$$P_1(x) \quad c_0 \quad c_1 x \qquad (10)$$

where c_0 and c_1 are constants, referred to as coefficients.
A polynomial of degree n is given by

$$P_n(x) \quad c_0 \quad c_1 x \quad c_2 x^2 \quad c_3 x^3 \qquad c_{n\ 1} x^{n\ 1} \quad c_n x^n \qquad (11)$$

It can be written compactly using the "sigma" or summation notation

$$P_n \ x) \quad \sum_{k\ 0}^{n} c_k x^k$$

$$P_n(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad (13)$$

Notice that the dummy index (m in this case) with summation notation is a compact way of writing the polynomial. Note that

$$P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a), \cdots, P_n^{(n)}(a) = f^{(n)}(a) \qquad (14)$$

The difference between $f(x)$ and $P_n(x)$ is called the remainder and is given by

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

other way around) if both mixed partial derivatives are continuous. This is generally the case in engineering applications. An alternate notation that you should be in which

will have dimensions of 1/time, and thus 1/

In engineering and physics, we often come up with the notion of a field variable. These are variables such as temperature, strain, pressure, etc., which are scalar quantities, and velocity, strain, shear stress, etc., which are either vectors or tensors in general. Think of a vector as a quantity that has both magnitude and direction, whereas a tensor is a quantity that has magnitude and two directions. Often, the second direction denotes the direction of the area associated with the tensor. Treating field variables as continuous functions of position and time allows for application of general principles to differential elements and to thereby

$$\boxed{\text{curl}\,\vec{F} = \nabla \times \vec{F}} \qquad (42)$$

which may be obtained by evaluating the determinant of the 3×3 matrix in the case of a RCC:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \hat{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \hat{j}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \qquad (43)$$

What are the dimensions of div and curl of a vector function? Note that determinants are reviewed in a Section K of this document.

Two integral theorems arise in engineering courses. The first is the divergence theorem according to which the volume integral of the divergence of a vector function $\vec{F}$

the solution vector. The goal is to find x. This kind of a problem arises very frequently in engineering.

Type 2:

$$Ax = \lambda x \text{ or written alternately as } (A - \lambda I)x = 0 \qquad (47)$$

where A is a square matrix of size $n \times n$, I is an identity matrix of the same size as A (i.e., I has 1's on the diagonal and zeros elsewhere), λ is an eigenvalue and the corresponding solution vector is referred to as an eigenvector. Note that the right hand side of (47) is strictly a column vector with all zero entries. The goal is to find the eigenvalues and corresponding eigenvectors. This type of problem arises less frequently, but is equally important for all engineering majors.

Determinant of a Matrix:

In either case, it is important to have a thorough understanding of properties of a matrix. Review concepts of matrix addition and matrix multiplication on your own. Also note that the transpose of matrix, denoted by A^T, is obtained by swapping the rows and columns of A, and a symmetric matrix is such that $A = A^T$. The most important property that is discussed here is that of a determinant, defined only for square matrices. In the case of a 2×2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad (48)$$

the determinant is defined by:

$$\boxed{\det A = |A| = ad - bc} \qquad (49)$$

This notion can be extended to matrices of size $n \times n$. For a general matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,n-1} & a_{nn} \end{bmatrix} \qquad (50)$$

The determinant is obtained by the method of cofactors. The cofactor of the i-th row and j-th column, denoted by M_{ij} is a square matrix of size $(n-1) \times (n-1)$ that is derived from A, by discarding its i-th row and j-th column. Using this definition, the determinant of A is obtained by:

$$\boxed{\det A = |A| = \sum_{i \text{ or } j} (-1)^{i+j} \left| M_{ij} \right|} \qquad (51)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (52)$$

by expanding using cofactors across the first row ($i = 1$)

$$\begin{aligned} \det A &= (-1)^{1+1} a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) + (-1)^{1+2} a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) \\ &\quad + (-1)^{1+3} a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right) \\ &= a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right) \end{aligned} \qquad (53)$$

or by expanding using cofactors down the first column ($j = 1$)

$$\begin{aligned} \det A &= (-1)^{1+1} a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) + (-1)^{2+1} a_{21}\left(a_{12}a_{33} - a_{32}a_{13}\right) \\ &\quad + (-1)^{3+1} a_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right) \\ &= a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{21}\left(a_{12}a_{33} - a_{32}a_{13}\right) + a_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right) \end{aligned} \qquad (54)$$

It is easy to verify that both approaches give the same numerical value for $\det A$ – in other words (53) and (54) are identical. Thus if a certain row or column has a number of zero entries, it makes sense to expand using cofactors for that row or column. Can you figure out why $\det A^T = \det A$? The concept of determinant is also helpful in finding the inverse of a square matrix, denoted by A^{-1}, such that $A^{-1}A = I$

As in the case of Type 1 problems, we will only discuss solution methods that work for small values of n (about 2 or 3), but become unwieldy for larger values of n. More efficient methods exist for finding eigenvalues and eigenvectors for larger size matrices that we will not worry about. To find the eigenvalues, find the characteristic equation by setting

$$\boxed{\det(A - \lambda I) = 0} \qquad (56)$$

This will be a polynomial of degree n. Refer to the section on polynomials to learn more about them. Using each of the computed eigenvalues λ_k, one can solve for the corresponding eigenvectors

$$Ax^{(k)} = \lambda_k x^{(k)} \qquad (57)$$

By setting an arbitrary, non-zero value for one of the components of $x^{(k)}$, the others can be found through the remaining set of consistent algebraic equations. Although the eigenvalues are unique, the eigenvectors are not – for instance an eigenvector corresponding to an eigenvalue can be scaled by multiplying it by a real constant, and it will still be an eigenvector. Usually, eigenvectors are reported in their normalized form. A normalized vector is obtained by dividing each of its components by the length or norm of the vector. The norm of a vector x is defined as $\sqrt{\sum_i x_i^2}$. Thus, the norm of a normalized vector is unity. Most computer programs report eigenvectors in normalized form. Note that in the case of a diagonal matrix (meaning $a_{ij} = 0, i \neq j$) or a triangular matrix, the eigenvalues are simply the entries on the diagonal.

Eigenvalues usually have a physical interpretation. In spring-mass systems subject to free (not forced) oscillations, the square root of an eigenvalue refers to the physical (angular) frequency of an oscillatory mode. The eigenvector may be interpreted as relative positions of the masses during oscillation in that particular mode. In solid mechanics, the stress tensor in three dimensions may be cast in the form of a 3×3 matrix that is symmetric. By choosing an appropriate coordinate system, the matrix can be transformed into a diagonal matrix whose entries refer to the principal components of stress. They are the eigenvalues of the stress tensor matrix. The eigenvectors are related to the direction cosines of the corresponding coordinate system.

L. Complex Numbers and Variables

The first time most students encounter complex numbers is in finding roots of a quadratic equation. Consider the algebraic equation:

$$ax^2 + bx + c = 0 \qquad (58)$$

The roots, or solutions x, that satisfy (58) are:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \qquad (59)$$

When the discriminant "$b^2 - 4ac$" is less than zero, the roots are complex. For example $\sqrt{-49} = \sqrt{-1}\sqrt{49} = 7i$, where i is defined by $\sqrt{-1}$

A complex number z is denoted by

$$z \quad x \quad i\,y \qquad (60)$$

in which x and y are real numbers, where x is denoted as the real part, $x \quad \mathrm{Re}(z)$, and y its imaginary part, $y \quad \mathrm{Im}(z)$. The conjugate of z (z as defined in (60)), denoted by $\bar{z}$, is defined as

$$\boxed{\bar{z} \quad x \quad i\,y} \qquad (61)$$

Note that when one root of a quadratic equation is complex, the other root is its complex conjugate, and thus roots of a quadratic (or for that matter a higher degree polynomial) always appear as complex conjugate pairs. The magnitude and argument of the complex number z, denoted by $|z|$ and $\arg(z)$

in which trigonometric identities are used in (68) and (69), and the trick to perform division, using the complex conjugate of the denominator to multiply both numerator and denominator, is used to get (69).

The basic algebra associated with complex numbers will serve you well for most purposes in undergraduate classes. Advanced concepts require the idea of complex variables and complex functions. Only the essential ideas are introduced here. A complex function $w = f(z)$ is given by

$$w = u + iv = f(z) = u(x, y) + iv(x, y) \qquad (70)$$

in which $u(x, y)$ and $v(x, y)$ are real and imaginary parts of the function, and are themselves, functions of two variables x and y. Note the subtle aspect of (70), in which the independent variable z in $w = f(z)$ is linked to the real and imaginary parts via x and y. Also note that (70) represents a mapping from the x-y or z plane to the u-v or w plane. This does not lend itself to a figure, quite as easily as functions of two independent variables in calculus of real functions. Perhaps the most important concept is that of an analytic function which we will not define formally. In very simple terms, it is a function that is well behaved without any singularities. Thus $w = z, w = z^2 + 3z$ are analytic functions, while $w = 1/z$ is analytic everywhere except at $z = 0$ where it is singular, and $1/(z-3)(z+2i)$ has singularities at $z = 3, z = -2i$. The exponential function is very important and is defined as:

$$\boxed{w = \exp z = e^{x+iy} = \exp(x)(\cos y + i \sin y)} \qquad (71)$$

and is analytic everywhere in the complex plane. It is defined in such a manner so that it reduces to the familiar $\exp(x)$ when the imaginary part of z, viz., y is identically zero. Without getting into details, it has the property that

$$\frac{d}{dz}\exp(z) = \exp(z) \qquad (72)$$

which is desirable when the imaginary part of z is zero and the result holds for $\exp(x)$. Similarly $\cos(z)$ and $\sin(z)$ are defined so that they become the familiar cosine and sine functions of real variables that we are familiar with when the imaginary part of z is identically zero.

$$\boxed{\cos z = \frac{\exp(iz) + \exp(-iz)}{2}; \; \sin z = \frac{\exp(iz) - \exp(-iz)}{2i}}$$

(73)

Defined in this manner, all trigonometric identities that you are familiar with, work fine for their complex counterparts (example: $\cos^2 z + \sin^2 z = 1$, etc.).